

# On the indefinite Kirchhoff type problems with local sublinearity and linearity\*

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## Abstract

The purpose of this paper is to study the indefinite Kirchhoff type problem:

$$\begin{cases} M \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx \right) [-\Delta u + u] = f(x, u) & \text{in } \mathbb{R}^N, \\ 0 \leq u \in H^1(\mathbb{R}^N), \end{cases}$$

where  $N \geq 1$ ,  $M(t) = am(t) + b$ ,  $m \in C(\mathbb{R}^+)$  and  $f(x, u) = g(x, u) + h(x)u^{q-1}$ .

We require that  $f$  is “local” sublinear at the origin and “local” linear at infinite.

Using the mountain pass theorem and Ekeland variational principle, the existence and multiplicity of nontrivial solutions are obtained. In particular, the criterion of existence of three nontrivial solutions is established.

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# 1 Introduction

In this paper, we investigate the existence and multiplicity of nontrivial solutions for a Kirchhoff type problem:

$$\begin{cases} M \left( \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx \right) [-\Delta u + u] = f(x, u) & \text{in } \mathbb{R}^N, \\ 0 \leq u \in H^1(\mathbb{R}^N), \end{cases} \quad (K)$$

where  $N \geq 1$ ,  $f \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$  and  $M : \mathbb{R} \rightarrow \mathbb{R}$  is a given function whose properties will be given later.

Problem (K) is related to the stationary analogue of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

presented by Kirchhoff [11] in 1883. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. Such problems are often referred to as being nonlocal because of the presence of the integral. When  $M(t) = at + b$  ( $a, b > 0$ ), it is degenerate if  $b = 0$  and nondegenerate otherwise.

After Lions [13] introduced an abstract framework to the Kirchhoff type problem, Problem (K) began to receive much attention. Most researchers studied the Kirchhoff type problems on bounded domain  $\Omega \subset \mathbb{R}^N$  with the following version

$$\begin{cases} -M \left( \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

For example, Bensedik and Boucekif [3], Chen et al. [4], Alves et al. [1] and Ma and Rivera [14], using variational methods, proved the existence and multiplicity of positive solutions while Zhang and Perera [19] obtained sign changing solutions via invariant sets of descent flow. In particular, Alves et al. [1] studied the conditions of  $M$  and  $f$  that permit the existence of a positive solution and concluded that this is possible if  $M$  does not grow too fast in a suitable interval near zero with  $f$  being locally Lipschitz subject to some prescribed criteria. Bensedik and Boucekif [3] studied the asymptotically linear case and obtained the existence of positive solutions of Problem (1) when the function  $M$  is a non-decreasing function and  $M \geq m_0$  for some  $m_0 > 0$ , and the assumptions about the asymptotic behaviors of  $f$  near zero and infinite are the following

- (f<sub>1</sub>)  $t \mapsto \frac{f(x,t)}{t}$  is a non-decreasing function for any fixed  $x \in \overline{\Omega}$ ;
- (f<sub>2</sub>)  $\lim_{t \rightarrow 0} \frac{f(x,t)}{t} = \overline{p}(x)$  and  $\lim_{t \rightarrow \infty} \frac{f(x,t)}{t} = \overline{q}(x)$  uniformly in  $x \in \Omega$ , where  $0 \leq \overline{p}(x), \overline{q}(x) \in L^\infty(\Omega)$  and  $\sup_{x \in \Omega} \overline{p}(x) < m_0 \lambda_1$ ,  $\lambda_1$  is the first eigenvalue of  $(-\Delta, H_0^1(\Omega))$ .

Compared with the case of the bounded domain  $\Omega \subset \mathbb{R}^N$ , the case of the whole space  $\mathbb{R}^N$  has been considered by a few authors, see [2, 5, 8, 10, 12, 15, 16, 17, 18], and the references therein. More precisely, Li et al. [12] considered the following Kirchhoff type problem:

$$(a + \lambda \int_{\mathbb{R}^N} (|\nabla u|^2 + bu^2) dx) [-\Delta u + bu] = f(u) \quad \text{in } \mathbb{R}^N, \quad (2)$$

where  $N \geq 3$ ,  $a$  and  $b$  are positive constants, and  $\lambda \geq 0$  is a parameter. Under the weaker assumption  $\lim_{t \rightarrow \infty} \frac{f(t)}{t} = \infty$ , a positive radial solution of Equation (2) was constructed by applying a monotonicity trick of Jeanjean [9] whenever  $\lambda \geq 0$  small enough. He and Zou [8] studied the multiplicity and concentration behavior of positive solutions for the following Kirchhoff type problem:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^N} |\nabla u| dx) \Delta u + V(x) u = f(u) & \text{in } \mathbb{R}^N, \\ 0 < u \in H^1(\mathbb{R}^N), \end{cases} \quad (3)$$

where  $\varepsilon > 0$  is a parameter,  $a, b > 0$  are constants, and  $f$  is a continuous superlinear and subcritical nonlinear term. When  $V$  has at least one minimum, the authors proved that Equation (3) has a ground state solution for  $\varepsilon > 0$  sufficiently small. Moreover, they investigated the relation between the number of positive solutions and the topology of the set of the global minima of the potentials by using minimax theorems together with the Ljusternik-Schnirelmann theory.

Inspired by the above facts, the aim of this paper is to consider the indefinite Kirchhoff type equations with local sublinearity and linearity. To the author's knowledge, this case seems to be considered by few authors. We mainly study the existence and multiplicity of nontrivial solutions for Problem (K). Furthermore, the non-existence of nontrivial solutions are also discussed. In this paper, we consider the following Kirchhoff type problem:

$$\begin{cases} M \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx \right) [-\Delta u + u] = g(x, u) + h(x) u^{q-1} & \text{in } \mathbb{R}^N, \\ 0 \leq u \in H^1(\mathbb{R}^N), \end{cases} \quad (K_{a,h})$$

where  $1 < q < 2$ ,  $h \in L^{2/(2-q)}(\mathbb{R}^N)$ ,  $M(t) = am(t) + b$ , the parameters  $a, b > 0$  and  $m$  is a continuous function on  $\mathbb{R}^+$  such that  $m(t) \geq 0$  for all  $t > 0$ . We assume that the function  $g$  satisfies the following conditions:

(D<sub>1</sub>)  $g(x, s)$  is a continuous function on  $\mathbb{R}^N \times \mathbb{R}$  such that  $g(x, s) \equiv 0$  for all  $s < 0$  and  $x \in \mathbb{R}^N$ . Moreover, there exists  $p_1 \in L^\infty(\mathbb{R}^N)$  with  $p_1^+ \not\equiv 0$  such that

$$\frac{g(x, s)}{s} \geq p_1 \quad \text{for all } s > 0 \text{ and } x \in \mathbb{R}^N$$

and

$$\lim_{s \rightarrow 0^+} \frac{g(x, s)}{s} = p_1 \quad \text{uniformly for } x \in \mathbb{R}^N,$$

where  $p_1^+ = \sup \{p_1, 0\}$ ;

(D<sub>2</sub>) there exists  $p_2 \in L^\infty(\mathbb{R}^N)$  with  $p_2^+ \not\equiv 0$  such that  $\lim_{s \rightarrow \infty} \frac{g(x,s)}{s} = p_2(x)$  uniformly for  $x \in \mathbb{R}^N$ , where  $p_2^+ = \sup \{p_2, 0\}$ ;

(D<sub>3</sub>)  $|p_1^+|_\infty < b < \frac{1}{\mu^*}$ , where

$$\mu^* := \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx : u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} p_2(x) u^2 dx = 1 \right\}; \quad (4)$$

(D<sub>4</sub>) there exists  $R_0 > 0$  such that

$$\sup \left\{ \frac{g(x,s)}{s} : s > 0 \right\} \leq \min \{1, b\} \text{ uniformly on } |x| \geq R_0.$$

**Remark 1.1** By conditions (D<sub>1</sub>) and (D<sub>2</sub>), the nonlinear term  $f(x, s) := g(x, s) + h(x) s^{q-1}$  for  $s > 0$  is “local” sublinear at the origin and “local” linear at infinite, i.e.

$$\lim_{s \rightarrow 0^+} \frac{f(x, s)}{s^{q-1}} = h^+(x) \text{ uniformly for } x \in \Omega_h^+ := \{x \in \mathbb{R}^N : h(x) > 0\}$$

and

$$\lim_{s \rightarrow \infty} \frac{f(x, s)}{s} = p_2^+(x) \text{ uniformly for } x \in \Omega_{p_2}^+ := \{x \in \mathbb{R}^N : p_2(x) > 0\},$$

where  $p^+ = \sup \{p, 0\}$ .

It is well known that Equation  $(K_{a,h})$  is variational and its solutions are the critical points of the functional defined in  $H^1(\mathbb{R}^N)$  by

$$I_{a,h}(u) = \frac{a}{2} \widehat{m}(\|u\|^2) + \frac{b}{2} \|u\|^2 - \int_{\mathbb{R}^N} G(x, u) dx - \int_{\mathbb{R}^N} h |u^+|^q dx,$$

where  $\widehat{m}(t) = \int_0^t m(s) ds$ ,  $\|u\| = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx \right)^{1/2}$  is a standard norm in  $H^1(\mathbb{R}^N)$ ,  $G(x, u) = \int_0^u g(x, s) ds$  and  $u^+ = \sup \{u, 0\}$ . Furthermore, it is easy to prove that the functional  $I_{a,h}$  is of class  $C^1$  in  $H^1(\mathbb{R}^N)$ , and that

$$\begin{aligned} \langle I'_{a,h}(u), v \rangle &= [am(\|u\|^2) + b] \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv) dx - \int_{\mathbb{R}^N} g(x, u) v dx \\ &\quad - \int_{\mathbb{R}^N} h |u^+|^{q-2} u^+ v dx. \end{aligned}$$

Hence if  $u \in H^1(\mathbb{R}^N)$  is a nonzero critical point of  $I_{a,h}$ , then  $u$  is a nontrivial solution of Equation  $(K_{a,h})$ .

Before stating our result we need to introduce some notations and definitions.

**Notation 1.1** Throughout this paper, we denote by  $|\cdot|_r$  the  $L^r$ -norm,  $2 \leq r \leq \infty$  and  $B_r := \{u \in H^1(\mathbb{R}^N) : \|u\| < r\}$  is an open ball in  $H^1(\mathbb{R}^N)$ . The letter  $C$  will denote various positive constants whose value may change from line to line but are not essential to the analysis of the problem. Also if we take a subsequence of a sequence  $\{u_n\}$  we shall denote it again  $\{u_n\}$ . We use  $o(1)$  to denote any quantity which tends to zero when  $n \rightarrow \infty$ .

**Definition 1.1**  $u$  is a ground state of Equation  $(K_{a,h})$  we mean that  $u$  is such a solution of Equation  $(K_{a,h})$  which has the least energy among all nontrivial solutions of Equation  $(K_{a,h})$ .

We also need the following assumptions:

(D<sub>5</sub>)  $m(t) \rightarrow +\infty$  as  $t \rightarrow \infty$ ;

(D<sub>6</sub>) there exist  $\delta_0, d_0 > 0$  such that  $m(t) \geq d_0 t^{\delta_0}$  for all  $t > 0$ .

Now, we give our main results.

**Theorem 1.1** (i) Suppose that conditions  $(D_1) - (D_5)$  hold. If  $h \equiv 0$ , then there exists  $a^* > 0$  such that for every  $a \in (0, a^*)$ , Equation  $(K_{a,h})$  has one nontrivial solution  $u_0^+$  with  $I_{a,h}(u_0^+) > 0$ .

(ii) Suppose that conditions  $(D_1) - (D_4)$  and  $(D_6)$  hold. If  $h \equiv 0$ , then there exists  $a^{**} > 0$  such that for every  $a \in (0, a^{**})$ , Equation  $(K_{a,h})$  has two nontrivial solutions  $u_0^-$  and  $u_0^+$  with  $I_{a,h}(u_0^-) < 0 < I_{a,h}(u_0^+)$ , and  $u_0^-$  is a ground state solution.

**Theorem 1.2** (i) Suppose that conditions  $(D_1) - (D_4)$  hold. Then there exists  $\Lambda_0 > 0$  such that for every  $a > 0$  and  $h \in L^{2/(2-q)}(\mathbb{R}^N)$  with  $0 < |h^+|_{L^{2/(2-q)}} < \Lambda_0$ , Equation  $(K_{a,h})$  has one nontrivial solution  $u_{h,1}^-$  with  $I_{a,h}(u_{h,1}^-) < 0$ .

(ii) Suppose that conditions  $(D_1) - (D_5)$  hold. Then there exist  $a^*, \Lambda_0 > 0$  such that for every  $a \in (0, a^*)$  and  $h \in L^{2/(2-q)}(\mathbb{R}^N)$  with  $0 < |h^+|_{L^{2/(2-q)}} < \Lambda_0$ , Equation  $(K_{a,h})$  has two nontrivial solutions  $u_{h,1}^-$  and  $u_h^+$  with  $I_{a,h}(u_{h,1}^-) < 0 < I_{a,h}(u_h^+)$ .

(iii) Suppose that conditions  $(D_1) - (D_4)$  and  $(D_6)$  hold. Then there exist  $\bar{a}_0, \bar{\Lambda}_0 > 0$  such that for every  $a \in (0, \bar{a}_0)$  and  $h \in L^{2/(2-q)}(\mathbb{R}^N)$  with  $h \geq 0$  and  $0 < |h|_{L^{2/(2-q)}} < \bar{\Lambda}_0$ , Equation  $(K_{a,h})$  has three nontrivial solutions  $u_{h,1}^-, u_{h,2}^-$  and  $u_h^+$  with

$$I_{a,h}(u_{h,2}^-) < I_{a,h}(u_{h,1}^-) < 0 < I_{a,h}(u_h^+),$$

and  $u_{h,2}^-$  is a ground state solution.

We now turn to example  $m(t) = t$  for  $t \geq 0$ .

**Corollary 1.3** *Suppose that conditions  $(D_1) - (D_4)$  hold and  $m(t) = t$  for  $t \geq 0$ . Then there exist  $\bar{a}_0, \bar{\Lambda}_0 > 0$  such that for every  $a \in (0, \bar{a}_0)$  and  $h \in L^{2/(2-q)}(\mathbb{R}^N)$  with  $h \geq 0$  and  $0 < |h|_{L^{2/(2-q)}} < \bar{\Lambda}_0$ , Equation  $(K_{a,h})$  has three nontrivial solutions  $u_{h,1}^-, u_{h,2}^-$  and  $u_h^+$  with*

$$I_{a,h}(u_{h,2}^-) < I_{a,h}(u_{h,1}^-) < 0 < I_{a,h}(u_h^+),$$

*and  $u_{h,2}^-$  is a ground state solution.*

**Remark 1.2** *If the functions  $p_1, p_2$  and  $h$  are nonnegative, then by the maximum principle, all solutions of Theorems 1.1 and 1.2 are positive ones of Equation  $(K_{a,h})$ .*

On the non-existence of nontrivial solutions we have the following result.

**Theorem 1.4** *Suppose in addition to the condition  $(D_2)$  holds and  $h \in L^{2/(2-q)}(\mathbb{R}^N)$ , we also have*

*$(D_7)$   $s \mapsto \frac{g(x,s)}{s}$  is non-decreasing function for any fixed  $x \in \mathbb{R}^N$ ;*

*$(D_8)$   $m(t) > \frac{1-b\mu^*}{a\mu^*} + \frac{|h^+|_{L^{2/(2-q)}}}{aS_2^q} t^{(q-2)/2}$  for all  $t > 0$ , where  $\mu^* > 0$  is defined in (4).*

*Then Equation  $(K_{a,h})$  does not admits any nontrivial solution.*

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3 and 4, we give the proofs of Theorems 1.1 and 1.2. In Section 5, we give the proof of Theorem 1.4.

## 2 Preliminaries

Throughout this paper, we denote by  $S_r$  the best Sobolev constant for the imbedding of  $H^1(\mathbb{R}^N)$  in  $L^r(\mathbb{R}^N)$  with  $2 \leq r < 2^*$ . In particular,

$$|u|_r \leq S_r^{-1} \|u\| \text{ for all } u \in H^1(\mathbb{R}^N) \setminus \{0\}.$$

Next, we give a useful theorem. It is the variant version of the mountain pass theorem, which allows us to find a so-called Cerami type  $(PS)$  sequence. The properties of this kind of  $(PS)$  sequence are very helpful in showing the boundedness of the sequence in the asymptotically linear case.

**Theorem 2.1** *([6], Mountain Pass Theorem). Let  $E$  be a real Banach space with its dual space  $E^*$ , and suppose that  $I \in C^1(E, \mathbb{R})$  satisfies*

$$\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} I(u),$$

for some  $\mu < \eta, \rho > 0$  and  $e \in E$  with  $\|e\| > \rho$ . Let  $c \geq \eta$  be characterized by

$$\alpha = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau)),$$

where  $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}$  is the set of continuous paths joining 0 and  $e$ , then there exists a sequence  $\{u_n\} \subset E$  such that

$$I(u_n) \rightarrow \alpha \geq \eta \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

In what follows, we give the following Lemmas which ensure that the functional  $I_{a,h}$  has the mountain pass geometry.

**Lemma 2.2** *Let  $1 < q < 2 < r, A > 0, B > 0$ , and consider the function*

$$\Psi_{A,B}(t) := t^2 - At^q - Bt^r \text{ for } t \geq 0.$$

*Then  $\max_{t \geq 0} \Psi_{A,B}(t) > 0$  if and only if*

$$A^{r-2}B^{2-q} < d(r, q) := \frac{(r-2)^{r-2}(2-q)^{2-q}}{(r-q)^{r-q}}.$$

*Furthermore, for  $t = t_B := [(2-q)/B(r-q)]^{1/(r-2)}$ , one has*

$$\Psi_{A,B}(t_B) = t_B^2 \left[ \frac{r-2}{r-q} - AB^{\frac{2-q}{r-2}} \left( \frac{r-q}{2-q} \right)^{\frac{2-q}{r-2}} \right] > 0.$$

**Proof.** The proof is essentially the same as that in [7, Lemma 3.2], and we omit it here.  $\blacksquare$

**Lemma 2.3** *Let  $1 < q < 2 < r < k, \bar{A} > 0, \bar{B} > 0$ , and consider the function*

$$\Phi_{\bar{A},\bar{B}}(t) := t^k - \bar{A}t^q - \bar{B}t^r \text{ for } t \geq 0.$$

*Then for  $t = t_{\bar{B}} := [\bar{B}(r-q)/(k-q)]^{1/(k-r)}$ , one has*

$$\Phi_{\bar{A},\bar{B}}(t_{\bar{B}}) = -t_{\bar{B}}^q \left[ \left( \frac{\bar{B}(r-q)}{k-q} \right)^{(k-q)/(k-r)} \left( \frac{k-r}{r-q} \right) + \bar{A} \right] < 0. \quad (5)$$

*Furthermore, there exist  $t_0, t_1 > 0$  such that  $\min_{t \geq 0} \Phi_{\bar{A},\bar{B}}(t) = \Phi_{\bar{A},\bar{B}}(t_0) < 0$  and  $\Phi_{\bar{A},\bar{B}}(t) \geq 0$  for all  $t \geq t_1$ .*

**Proof.** Since  $\Phi_{\bar{A},\bar{B}}(t) = t^q (t^{k-q} - \bar{A} - \bar{B}t^{r-q})$ , it follows that  $\Phi_{\bar{A},\bar{B}}(t) < 0$  if and only if  $t^{k-q} - \bar{A} - \bar{B}t^{r-q} < 0$ . The derivative of  $t^{k-q} - \bar{A} - \bar{B}t^{r-q}$  vanishes exactly for  $t = t_{\bar{B}}$  and one readily computes  $\Phi_{\bar{A},\bar{B}}(t_{\bar{B}})$ , as indicated in (5). The conclusion of the lemma then follows easily. ■

**Lemma 2.4** *Let  $2 < r < k$ ,  $A_0 > 0$ ,  $B_0 > 0$ , and consider the function*

$$\Theta_{A_0,B_0}(t) := t^k + A_0 t^2 - B_0 t^r \text{ for } t \geq 0.$$

*Then  $\min_{t \geq 0} \Theta_{A_0,B_0}(t) < 0$  if and only if*

$$A_0^{k-r} B_0^{2-k} < d_0(r) := \frac{(k-r)^{k-r} (r-2)^{r-2}}{(k-2)^{k-2}}.$$

*For  $t = t_{B_0} := [B_0(r-2)/(k-2)]^{1/(k-r)}$ , one has*

$$\Theta_{A_0,B_0}(t_{B_0}) = t_{B_0}^2 \left[ A_0 - B_0 \left( \frac{B_0(r-2)}{k-2} \right)^{\frac{r-2}{k-r}} \left( \frac{k-r}{k-2} \right) \right] < 0. \quad (6)$$

*Furthermore, there exist  $t_0 < t_{B_0} < t_1$  such that  $\Theta_{A_0,B_0}(t_0) = \Theta_{A_0,B_0}(t_1) = 0$  and  $\Theta_{A_0,B_0}(t) > 0$  for all  $t \in (0, t_0) \cup (t_1, \infty)$ .*

**Proof.** Since  $\Theta_{A_0,B_0}(t) = t^2 (t^{k-2} + A_0 - B_0 t^{r-2})$ , it follows that  $\Theta_{A_0,B_0}(t) < 0$  if and only if  $t^{k-2} + A_0 - B_0 t^{r-2} < 0$ . The derivative of  $t^{k-2} + A_0 - B_0 t^{r-2}$  vanishes exactly for  $t = t_{B_0}$  and one readily computes  $\Theta_{A_0,B_0}(t_{B_0})$ , as indicated in (6). The conclusion of the lemma then follows easily. ■

**Lemma 2.5** *Suppose that conditions  $(D_1) - (D_3)$  hold. Then there exist  $\Lambda_0, \rho > 0$  such that for every  $h \in L^{2/(2-q)}(\mathbb{R}^N)$  with  $|h^+|_{L^{2/(2-q)}} < \Lambda_0$ ,*

$$\inf \{ I_{a,h}(u) : u \in H^1(\mathbb{R}^N) \text{ with } \|u\| = \rho \} > \eta$$

*for some  $\eta > 0$ . Furthermore, if  $h^+ \equiv 0$ , then there exists  $\rho_0 > \rho$  such that  $I_{a,h}(u) > 0$  for all  $u \in B_{\rho_0} \setminus \{0\}$  and  $\inf_{u \in B_{\rho_0}} I_{a,h}(u) = 0$ .*

**Proof.** By conditions  $(D_1) - (D_3)$  and  $(D_5)$ , and noticing that  $\lim_{s \rightarrow +\infty} \frac{g(x,s)}{s^{r-1}} = 0$  uniformly in  $x \in \mathbb{R}^N$  for any fixed  $2 < r < 2^*$  ( $2^* = \infty$  if  $N = 1, 2$  and  $2^* = \frac{2N}{N-2}$  if  $N \geq 3$ ), it is easy to see that for every  $\epsilon > 0$ , there exists  $C_\epsilon = C(\epsilon, r, g) > 0$  such that

$$g(x, s) \leq \frac{|p_1^+|_\infty + \epsilon}{2} s + \frac{C_\epsilon}{r} |s|^{r-1}, \quad \text{for all } s \geq 0 \quad (7)$$



and

$$G(x, s) \leq \frac{|p_1^+|_\infty + \epsilon}{2} s^2 + \frac{C_\epsilon}{r} |s|^r, \quad \text{for all } s \geq 0.$$

Since  $|p_1^+|_\infty < b$ , we can find  $\epsilon_0 > 0$  with  $|p_1^+|_\infty + \epsilon_0 < b$  and there is a  $C_0 = C(\epsilon_0, r, g) > 0$  such that

$$G(x, s) \leq \frac{|p_1^+|_\infty + \epsilon_0}{2} s^2 + \frac{C_0}{r} |s|^r, \quad \text{for all } s \geq 0. \quad (8)$$

Thus, from (8) and the Sobolev inequality, we have for all  $u \in H^1(\mathbb{R}^N)$ ,

$$\begin{aligned} \int_{\mathbb{R}^N} G(x, u) dx &\leq \frac{|p_1^+|_\infty + \epsilon_0}{2} \int_{\mathbb{R}^N} u^2 dx + \frac{C_0}{r} \int_{\mathbb{R}^N} |u|^r dx \\ &\leq \frac{|p_1^+|_\infty + \epsilon_0}{2} \|u\|^2 + \frac{C_0 S_r^{-r}}{r} \|u\|^r, \end{aligned} \quad (9)$$

which implies that

$$\begin{aligned} I_{a,h}(u) &= \frac{a}{2} \widehat{m}(\|u\|^2) + \frac{b}{2} \|u\|^2 - \int_{\mathbb{R}^N} G(x, u) dx - \frac{1}{q} \int_{\mathbb{R}^N} h |u^+|^q dx \\ &\geq \frac{b}{2} \|u\|^2 - \frac{|p_1^+|_\infty + \epsilon_0}{2} \|u\|^2 - \frac{C_0 S_r^{-r}}{r} \|u\|^r - \frac{|h^+|_{L^{2/(2-q)}}}{q S_2^q} \|u\|^q \\ &\geq \frac{b - |p_1^+|_\infty - \epsilon_0}{2} \|u\|^2 - \frac{C_0}{r S_r^r} \|u\|^r - \frac{|h^+|_{L^{2/(2-q)}}}{q S_2^q} \|u\|^q \end{aligned} \quad (10)$$

for all  $u \in H^1(\mathbb{R}^N)$ . For  $h \in L^{2/(2-q)}(\mathbb{R}^N)$  with  $|h^+|_{L^{2/(2-q)}} > 0$ . We now apply to Lemma 2.2 above with

$$A = \frac{2 |h^+|_{L^{2/(2-q)}}}{q S_2^q (b - |p_1^+|_\infty - \epsilon_0)} > 0 \text{ and } B = \frac{2 C_0}{r S_r^r (b - |p_1^+|_\infty - \epsilon_0)} > 0.$$

This shows that for all  $u \in H^1(\mathbb{R}^N)$  with  $\|u\| = t_B = [(2 - q) / B (r - q)]^{1/(r-2)}$ ,

$$I_{a,h}(u) \geq \frac{b - |p_1^+|_\infty - \epsilon_0}{2} \Psi_{A,B}(t_B) > 0$$

provided that  $A^{r-2} B^{2-q} < d(r, s)$ , i.e., provided that

$$|h^+|_{L^{2/(2-q)}} < \Lambda_0 := \frac{(r-2) S_2^q}{2} \left( \frac{b - |p_1^+|_\infty - \epsilon_0}{r - s} \right)^{(r-q)/(r-2)} \left( \frac{r S_r^r (2 - s)}{2 C_0} \right)^{(2-q)/(r-2)}.$$

Letting

$$\rho = t_B = [(2 - q) / B (r - q)]^{1/(r-2)} > 0$$

and

$$\eta = \frac{b - |p_1^+|_\infty - \epsilon_0}{2} \Psi_{A,B}(t_B) > 0,$$

it is easy to see that the result holds. Moreover, if  $h^+ \equiv 0$ , then by (10),

$$I_{a,h}(u) > 0 \text{ for all } u \in B_{\rho_0} \setminus \{0\}$$

and

$$\inf_{u \in B_{\rho_0}} I_{a,h}(u) = 0,$$

where  $\rho_0 = \left(\frac{r-q}{2-r}\right)^{1/(r-2)} \rho > \rho$ . This completes the proof. ■

**Lemma 2.6** *Suppose that conditions  $(D_1) - (D_3)$  hold. Then for each  $h \in L^{2/(2-q)}(\mathbb{R}^N)$  there exist  $a^* > 0$  and  $e \in H^1(\mathbb{R}^N)$  with  $\|e\| > \rho$  such that  $I_{a,h}(e) < 0$  for all  $a \in (0, a^*)$ , where  $\rho$  is given by Lemma 2.5.*

**Proof.** By the condition  $(D_3)$ , in view of the definition of  $\mu^*$  and  $b < 1/\mu^*$ , there is  $\phi \in H^1(\mathbb{R}^N) \setminus \{0\}$  with  $\phi \geq 0$  such that  $\int_{\mathbb{R}^N} p_2(x) \phi^2 dx = 1$  and  $b\mu^* \leq b\|\phi\|^2 < 1$ . According to the condition  $(D_2)$  and Fatou's lemma, we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{I_{0,h}(t\phi)}{t^2} &= \frac{b}{2} \|\phi\|^2 - \lim_{t \rightarrow +\infty} \int_{\mathbb{R}^N} \frac{G(x, t\phi)}{t^2 \phi^2} \phi^2 dx - \lim_{t \rightarrow \infty} \frac{1}{t^{2-q}} \int_{\mathbb{R}^N} h(x) |\phi|^q dx \\ &\leq \frac{b}{2} \|\phi\|^2 - \int_{\mathbb{R}^N} \lim_{t \rightarrow +\infty} \frac{G(x, t\phi)}{t^2 \phi^2} \phi^2 dx \\ &= \frac{b}{2} \|\phi\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} p_2(x) \phi^2 dx \\ &= \frac{1}{2} (b\|\phi\|^2 - 1) < 0, \end{aligned}$$

where  $I_{0,h} = I_{a,h}$  for  $a = 0$ . So, if  $I_{0,h}(t\phi) \rightarrow -\infty$  as  $t \rightarrow +\infty$ , then there exists  $e \in H^1(\mathbb{R}^N)$  with  $\|e\| > \rho$  such that  $I_{0,h}(e) < 0$ . Since  $I_{a,h}(e) \rightarrow I_{0,h}(e)$  as  $a \rightarrow 0^+$ , there exists  $a^* > 0$  such that  $I_{a,h}(e) < 0$  for all  $a \in (0, a^*)$  and the lemma is proved. ■

**Lemma 2.7** *Suppose that conditions  $(D_1) - (D_3)$  and  $(D_6)$  hold. Let  $h \in L^{2/(2-q)}(\mathbb{R}^N)$  and  $a^* > 0$  be as in Lemma 2.6. Then for every  $a \in (0, a^*)$  there exists  $D_a < 0$  such that*

$$D_a \leq \tilde{\theta}_h := \inf \{I_{a,h}(u) : u \in H^1(\mathbb{R}^N)\} < 0.$$

Furthermore, there exists  $R_h > 0$  such that  $I_{a,h}(u) > 0$  for all  $u \in H^1(\mathbb{R}^N)$  with  $\|u\| \geq R_h$ , and

$$\inf \{I_{a,h}(u) : u \in H^1(\mathbb{R}^N)\} = \inf \{I_{a,h}(u) : u \in B_{R_h}\} < 0.$$

In particular,

$$\inf \{I_{a,0}(u) : u \in H^1(\mathbb{R}^N)\} = \inf \{I_{a,0}(u) : u \in B_{R_h}\} < 0,$$

where  $I_{a,0}(u) = I_{a,h}(u)$  for  $h \equiv 0$ .

**Proof.** By conditions  $(D_1) - (D_3)$  and  $(D_6)$ , and noticing that  $\lim_{s \rightarrow +\infty} \frac{g(x,s)}{s^{r-1}} = 0$  uniformly in  $x \in \mathbb{R}^N$  for any fixed  $2 < r < \min\{2 + \delta_0, 2^*\}$  ( $2^* = \infty$  if  $N = 1, 2$  and  $2^* = \frac{2N}{N-2}$  if  $N \geq 3$ ), it is easy to see that for every  $\epsilon_0 > 0$  with  $|p_1^+|_\infty + \epsilon_0 < b$ , there is a  $C_0 = C(\epsilon_0, r, g) > 0$  such that

$$\begin{aligned} I_{a,h}(u) &= \frac{a}{2} \widehat{m}(\|u\|^2) + \frac{b}{2} \|u\|^2 - \int_{\mathbb{R}^N} G(x, u) dx - \int_{\mathbb{R}^N} h |u^+|^q dx \\ &\geq \frac{ad_0}{2 + 2\delta_0} \|u\|^{2+2\delta_0} + \frac{b - |p_1^+|_\infty - \epsilon_0}{2} \|u\|^2 - \frac{C_0}{rS_r} \|u\|^r - S_2^{-q} |h^+|_{L^{2/(2-q)}} \|u\|^q \\ &\geq \frac{ad_0}{2 + 2\delta_0} \|u\|^{2+2\delta_0} - \frac{C_0}{rS_r} \|u\|^r - S_2^{-q} |h^+|_{L^{2/(2-q)}} \|u\|^q \end{aligned}$$

for all  $u \in H^1(\mathbb{R}^N)$ , where we have used (9) and the condition  $(D_6)$ . Then

$$I_{a,h}(u) \geq \frac{ad_0}{2 + 2\delta_0} \left( \|u\|^{2+2\delta_0} - \frac{(2 + 2\delta_0) |h^+|_{L^{2/(2-q)}}}{ad_0 S_2^q} \|u\|^q - \frac{(2 + 2\delta_0) C_0}{ard_0 S_r^r} \|u\|^r \right).$$

We now apply to Lemma 2.3 above with

$$\overline{A} = \frac{(2 + 2\delta_0) |h^+|_{L^{2/(2-q)}}}{ad_0 S_2^q} > 0 \text{ and } \overline{B} = \frac{(2 + 2\delta_0) C_0}{ard_0 S_r^r} > 0,$$

then for every  $a \in (0, a^*)$ , there exist  $D_a < 0$  and  $R_h > t_{\overline{B}} = [\overline{B}(r - q)/(k - q)]^{1/(k-r)}$  such that

$$I_{a,h}(u) \geq D_a \text{ for all } u \in H^1(\mathbb{R}^N),$$

and

$$I_{a,h}(u) > 0 \text{ for all } u \in H^1(\mathbb{R}^N) \text{ with } \|u\| \geq R_h, \quad (11)$$

which implies that

$$\inf \{I_{a,h}(u) : u \in H^1(\mathbb{R}^N)\} \geq D_a.$$

Moreover, by Lemma 2.6, for any  $a \in (0, a^*)$

$$\inf \{I_{a,h}(u) : u \in H^1(\mathbb{R}^N)\} < 0. \quad (12)$$

Therefore, combining (11) and (12), we see that

$$\inf \{I_{a,h}(u) : u \in H^1(\mathbb{R}^N)\} = \inf \{I_{a,h}(u) : u \in \overline{B}_{R_h}\} < 0.$$

This completes the proof. ■

By Theorem 2.1 and Lemmas 2.5, 2.6, we obtain that there is a sequence  $\{u_n\} \subset H^1(\mathbb{R}^N)$  such that

$$I_{a,h}(u_n) \rightarrow \alpha > 0 \quad \text{and} \quad (1 + \|u_n\|) \|I'_{a,h}(u_n)\|_{H^{-1}(\mathbb{R}^N)} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (13)$$

**Lemma 2.8** *Suppose that conditions  $(D_1)$ ,  $(D_2)$  and  $(D_5)$  hold. Then  $\{u_n\}$  defined in (13) is bounded in  $H^1(\mathbb{R}^N)$ .*

**Proof.** By contradiction, let  $\|u_n\| \rightarrow +\infty$  as  $n \rightarrow \infty$ . Define  $w_n := \frac{u_n}{\|u_n\|}$ . Clearly,  $w_n$  is bounded in  $H^1(\mathbb{R}^N)$  and there is  $w \in H^1(\mathbb{R}^N)$  such that, up to a subsequence,

$$w_n \rightharpoonup w \text{ weakly in } H^1(\mathbb{R}^N) \text{ and } w_n \rightarrow w \text{ strongly in } L^2_{loc}(\mathbb{R}^N) \text{ as } n \rightarrow \infty.$$

It follows from (13) that

$$\frac{\langle I'_{a,h}(u_n), u_n \rangle}{\|u_n\|^2} = o(1),$$

that is,

$$o(1) = am(\|u_n\|^2) + b - \int_{\mathbb{R}^N} \frac{g(x, u_n)}{u_n} w_n^2 dx - \frac{\int_{\mathbb{R}^N} h(x) |w_n^+|^q dx}{\|u_n\|^{2-q}}, \quad (14)$$

where  $o(1)$  denotes a quantity which goes to zero as  $n \rightarrow \infty$  and  $\delta_0$  is as in the condition  $(D_5)$ . By conditions  $(D_1)$  and  $(D_2)$  and  $h \in L^{2/(2-q)}(\mathbb{R}^N)$ , there exist  $C_1, C_2 > 0$  such that

$$\frac{g(x, s)}{s} \leq C_1 \text{ for all } s \in \mathbb{R}$$

and

$$\int_{\mathbb{R}^N} h(x) |w_n^+|^q dx \leq C_2 \|h^+\|_{L^{2/(2-q)}},$$

which implies that  $\int_{\mathbb{R}^N} \frac{g(x, u_n)}{u_n} w_n^2 dx$  and  $\int_{\mathbb{R}^N} h(x) |w_n^+|^q dx$  are bounded in  $H^1(\mathbb{R}^N)$ . Moreover, by the condition  $(D_5)$ ,

$$m(\|u_n\|^2) \rightarrow +\infty \text{ as } n \rightarrow \infty.$$

So the above equation (14) is a contradiction. Therefore,  $\{u_n\}$  is bounded in  $H^1(\mathbb{R}^N)$ . This completes the proof. ■

### 3 Proof of Theorem 1.1

To prove that the Cerami sequence  $\{u_n\}$  in (13) converges to a nonzero critical point of  $I_{a,h}$ , the following compactness lemma is useful.

**Lemma 3.1** *Suppose that conditions  $(D_1) - (D_4)$  hold and  $\{u_n\}$  is a  $(PS)_\beta$ -sequence for  $I_{a,h}$  in  $H^1(\mathbb{R}^N)$ , that is*

$$I_{a,h}(u_n) \rightarrow \beta \quad \text{and} \quad \|I'_{a,h}(u_n)\|_{H^{-1}(\mathbb{R}^N)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

*If  $\{u_n\}$  is a bounded sequence in  $H^1(\mathbb{R}^N)$ , then for any  $\epsilon > 0$ , there exist  $R(\epsilon) > R_0$  and  $n(\epsilon) > 0$  such that  $\int_{|x| \geq R} (|\nabla u_n|^2 + u_n^2) dx \leq \epsilon$  for all  $n \geq n(\epsilon)$  and  $R \geq R(\epsilon)$ .*

**Proof.** Let  $\xi_R : \mathbb{R}^3 \rightarrow [0, 1]$  be a smooth function such that

$$\xi_R(x) = \begin{cases} 0, & 0 \leq |x| \leq R, \\ 1, & |x| \geq 2R, \end{cases} \quad (15)$$

and for some constant  $C > 0$  (independent of  $R$ ),

$$|\nabla \xi_R(x)| \leq \frac{C}{R}, \quad \text{for all } x \in \mathbb{R}^3. \quad (16)$$

Then, for all  $n \in \mathbb{N}$  and  $R \geq R_0$ , we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla(u_n \xi_R)|^2 dx &= \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R^2 dx + \int_{\mathbb{R}^N} |u_n|^2 |\nabla \xi_R|^2 dx \\ &\leq \int_{R < |x| < 2R} |\nabla u_n|^2 dx + \int_{|x| > 2R} |\nabla u_n|^2 dx + \frac{C^2}{R^2} \int_{\mathbb{R}^N} u_n^2 dx \\ &\leq (2 + \frac{C^2}{R^2}) \|u_n\|^2 \leq (2 + \frac{C^2}{R_0^2}) \|u_n\|^2. \end{aligned}$$

This implies that

$$\|u_n \xi_R\| \leq (3 + \frac{C^2}{R_0^2})^{\frac{1}{2}} \|u_n\|, \quad \text{for all } n \in \mathbb{N} \text{ and } R \geq R_0. \quad (17)$$

Since  $\{u_n\}$  is a bounded sequence in  $H^1(\mathbb{R}^N)$ , it follows that  $\|I'_{a,h}(u_n)\|_{H^{-1}} \|u_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . So, for any  $\epsilon > 0$ , there exists  $n(\epsilon) > 0$  such that

$$\|I'_{a,h}(u_n)\|_{H^{-1}} \|u_n\| \leq \epsilon (3 + \frac{C^2}{R_0^2})^{-\frac{1}{2}}, \quad \text{for all } n > n(\epsilon). \quad (18)$$

Hence, it follows from (17) and (18) that

$$|\langle I'_{a,h}(u_n), u_n \xi_R \rangle| \leq \|I'_{a,h}(u_n)\|_{H^{-1}} \|u_n \xi_R\| \leq \epsilon, \quad (19)$$

for all  $n > n(\epsilon)$  and  $R > R_0$ . Note that

$$\begin{aligned}
& \langle I'_{a,h}(u_n), u_n \xi_R \rangle \\
&= am(\|u_n\|^2) \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + \int_{\mathbb{R}^N} u_n^2 \xi_R dx + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \xi_R dx \right) \\
&+ b \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + \int_{\mathbb{R}^N} u_n^2 \xi_R dx + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \xi_R dx \right) \\
&- \int_{\mathbb{R}^N} g(x, u_n) u_n \xi_R dx - \int_{\mathbb{R}^N} h |u_n^+|^q \xi_R dx \\
&\geq b \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + \int_{\mathbb{R}^N} u_n^2 \xi_R dx \right) + (am(\|u_n\|^2) + b) \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \xi_R dx \\
&- \int_{\mathbb{R}^N} g(x, u_n) u_n \xi_R dx - \int_{\mathbb{R}^N} h |u_n^+|^q \xi_R dx.
\end{aligned} \tag{20}$$

For any  $\epsilon > 0$ , there exists  $R_1(\epsilon) > R_0$  such that

$$\frac{1}{R^2} \leq \frac{4\epsilon^2}{C^2} \text{ for all } R \geq R(\epsilon). \tag{21}$$

By (21) and the Young inequality, we get, for all  $n \in \mathbb{N}$  and  $R \geq R(\epsilon)$ ,

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_n \nabla u_n \nabla \xi_R| dx &\leq \epsilon \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{4\epsilon} \int_{|x| \leq 2R} |u_n|^2 \frac{C^2}{R^2} dx \\
&\leq \epsilon \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \epsilon \int_{|x| \leq 2R} |u_n|^2 dx \\
&\leq \epsilon \|u_n\|^2.
\end{aligned} \tag{22}$$

Moreover, since  $h \in L^{2/(2-q)}(\mathbb{R}^N)$ , using the Egorov theorem and the Hölder inequality, there exists  $R_2(\epsilon) > R_0$  such that for all  $n \in \mathbb{N}$  and  $R \geq R_2(\epsilon)$ , we have

$$\int_{\mathbb{R}^N} h |u_n^+|^q \xi_R < \epsilon. \tag{23}$$

Take  $R(\epsilon) = \max\{R_1(\epsilon), R_2(\epsilon)\}$ . Now we consider two cases.

**Case (i):**  $b \geq 1$ . By  $(D_1), (D_4)$  and (16), there exists  $\eta_1 \in (0, 1)$  such that, for all  $n \in \mathbb{N}$  and  $R \geq R_0$ ,

$$\int_{\mathbb{R}^N} |g(x, u_n) u_n \xi_R| dx \leq \eta_1 \int_{\mathbb{R}^N} u_n^2 \xi_R dx.$$

Using this, together with (20), (22) and (23), for all  $n \in \mathbb{N}$  and  $R \geq R(\epsilon) \geq R_0$ , we see that

$$\begin{aligned}
& \langle I'_{a,h}(u_n), u_n \xi_R \rangle \\
& \geq b \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + \int_{\mathbb{R}^N} u_n^2 \xi_R dx \right) \\
& \quad + (am(\|u_n\|^2) + b) \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \xi_R dx - \int_{\mathbb{R}^N} g(x, u_n) u_n \xi_R dx - \int_{\mathbb{R}^N} h |u_n^+|^q \xi_R \\
& \geq \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + \int_{\mathbb{R}^N} u_n^2 \xi_R dx + (am(\|u_n\|^2) + 1) \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \xi_R dx \\
& \quad - \int_{\mathbb{R}^N} g(x, u_n) u_n \xi_R dx - \int_{\mathbb{R}^N} h |u_n^+|^q \xi_R \\
& \geq \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + (1 - \eta_1) \int_{\mathbb{R}^N} u_n^2 \xi_R dx - \epsilon [(am(\|u_n\|^2) + 1) \|u_n\|^2 + 1]. \quad (24)
\end{aligned}$$

Since  $\{u_n\}$  is bounded in  $H^1(\mathbb{R}^N)$ , it follows from (19) and (24) that there exists  $C_2 > 0$  such that for all  $n \geq n(\epsilon)$  and  $R \geq R(\epsilon)$ ,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + (1 - \eta_1) \int_{\mathbb{R}^N} u_n^2 \xi_R dx \leq C_2 \epsilon. \quad (25)$$

From  $\eta_1 \in (0, 1)$  and (15), it is easy to see that (25) implies the final conclusion.

**Case (ii)**  $0 < b < 1$ . By  $(D_1)$ ,  $(D_4)$  and (16), there exists  $\eta_2 \in (0, 1)$  such that, for all  $n \in \mathbb{N}$  and  $R \geq R_0$ ,

$$\int_{\mathbb{R}^N} |g(x, u_n) u_n \xi_R| dx \leq b \eta_2 \int_{\mathbb{R}^N} u_n^2 \xi_R dx.$$

Similar to the proof of Case (i), we have

$$\begin{aligned}
& \langle I'_{a,h}(u_n), u_n \xi_R \rangle \\
& \geq b \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + \int_{\mathbb{R}^N} u_n^2 \xi_R dx \right) \\
& \quad + [am(\|u_n\|^2) + b] \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \xi_R dx - \int_{\mathbb{R}^N} g(x, u_n) u_n \xi_R dx - \int_{\mathbb{R}^N} h |u_n^+|^q \xi_R \\
& \geq b \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + b(1 - \eta_2) \int_{\mathbb{R}^N} u_n^2 \xi_R dx \\
& \quad - \epsilon [(am(\|u_n\|^2) + b) \|u_n\|^2 + 1],
\end{aligned}$$

and there exists  $C_3 > 0$  such that for all  $n \geq n(\epsilon)$  and  $R \geq R(\epsilon)$ ,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R dx + (1 - \eta_2) \int_{\mathbb{R}^N} u_n^2 \xi_R dx \leq C_3 \epsilon. \quad (26)$$

From  $\eta_2 \in (0, 1)$  and (15), it is easy to see that (26) also implies the final conclusion. ■

**Lemma 3.2** *Suppose that conditions  $(D_1) - (D_4)$  hold. Let  $\{u_n\}$  be a sequence as in (13). Then for any  $\epsilon > 0$ , there exist  $R(\epsilon) > R_0$  and  $n(\epsilon) > 0$  such that  $\int_{|x| \geq R} (|\nabla u_n|^2 + u_n^2) dx \leq \epsilon$  for all  $n \geq n(\epsilon)$  and  $R \geq R(\epsilon)$ .*

**Proof.** Clearly,  $\{u_n\}$  is a  $(PS)_\alpha$ -sequence for  $I_{a,h}$  in  $H^1(\mathbb{R}^N)$ . Moreover, by Lemma 2.8,  $\{u_n\}$  is a bounded sequence in  $H^1(\mathbb{R}^N)$ . Thus, by Lemma 3.1, it is easy to see that this lemma holds. ■

**Theorem 3.3** *Suppose that conditions  $(D_1) - (D_5)$  hold and  $h \in L^{2/(2-q)}(\mathbb{R}^N)$ . Let  $a^* > 0$  be as in Lemma 2.6. Then for each  $a \in (0, a^*)$ ,  $I_{a,h}$  has a nonzero critical point  $u_0 \in H^1(\mathbb{R}^3)$  such that  $I_{a,h}(u_0) = \alpha > 0$ .*

**Proof.** By Lemma 2.8, the sequence  $\{u_n\}$  in (13) is bounded in  $H^1(\mathbb{R}^N)$ . We may assume that, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u_0 \text{ weakly in } H^1(\mathbb{R}^N); \\ u_n &\rightarrow u_0 \text{ strongly in } L^r_{loc}(\mathbb{R}^N) \text{ for } 2 \leq r < 2^*; \\ u_n &\rightarrow u_0 \text{ a.e. in } \mathbb{R}^N \end{aligned} \tag{27}$$

for some  $u_0 \in H^1(\mathbb{R}^N)$ . Moreover, since  $h \in L^{2/(2-q)}(\mathbb{R}^N)$ , using the Egorov theorem and the Hölder inequality, we have

$$\int_{\mathbb{R}^N} h |u_n^+|^{q-2} u_n^+ (u_n^+ - u_0^+) = o(1). \tag{28}$$

In order to prove our conclusion, it is now sufficient to show that  $\|u_n\| \rightarrow \|u_0\|$  as  $n \rightarrow \infty$ . Note that, by (13),

$$\begin{aligned} \langle I'_{a,h}(u_n), u_n \rangle &= [am(\|u_n\|^2) + b] \int_{\mathbb{R}^N} (|\nabla u_n|^2 + u_n^2) dx \\ &\quad - \int_{\mathbb{R}^N} g(x, u_n) u_n dx - \int_{\mathbb{R}^N} h(x) |u_n^+|^q dx \\ &= o(1), \end{aligned}$$

and

$$\begin{aligned} \langle I'_{a,h}(u_n), u_0 \rangle &= [am(\|u_n\|^2) + b] \int_{\mathbb{R}^N} (\nabla u_n \nabla u_0 + u_n u_0) dx \\ &\quad - \int_{\mathbb{R}^N} g(x, u_n) u_0 dx - \int_{\mathbb{R}^N} h(x) |u_n^+|^{q-2} u_n^+ u_0^+ dx \\ &= o(1). \end{aligned}$$



Since  $u_n \rightharpoonup u_0$  weakly in  $H^1(\mathbb{R}^N)$ , it follows that

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla u_0 + u_n u_0) dx = \int_{\mathbb{R}^N} (|\nabla u_0|^2 + u_0^2) dx + o(1). \quad (29)$$

So by (28) and (29), to show  $\|u_n\| \rightarrow \|u_0\|$  is equivalent to prove that

$$\int_{\mathbb{R}^N} g(x, u_n) u_n dx = \int_{\mathbb{R}^N} g(x, u_n) u_0 dx + o(1). \quad (30)$$

Indeed, for any  $\epsilon > 0$ , by the condition  $(D_4)$ , the Hölder inequality and Lemma 3.2, for  $n$  large enough, one has

$$\begin{aligned} & \int_{|x| \geq R(\epsilon)} g(x, u_n) u_n dx - \int_{|x| \geq R(\epsilon)} g(x, u_n) u_0 dx \\ & \leq \int_{|x| \geq R(\epsilon)} |g(x, u_n)| |u_n - u_0| dx \\ & \leq \min \{1, b\} \int_{|x| \geq R(\epsilon)} |u_n| |u_n - u_0| dx \\ & \leq \min \{1, b\} \left( \int_{|x| \geq R(\epsilon)} |u_n|^2 dx \right)^{\frac{1}{2}} \left( \int_{|x| \geq R(\epsilon)} |u_n - u_0|^2 dx \right)^{\frac{1}{2}} \\ & \leq \min \{1, b\} \epsilon. \end{aligned}$$

Combining this and (27), the equation (30) holds. This completes the proof. ■

**Now we give the proof of Theorem 1.1:** (i) Theorem 3.3 shows the conclusion. (ii) By Lemma 2.7 and the Ekeland variational principle, there exists a minimizing sequence  $\{u_n\} \subset B_{R_h}$  such that

$$I_{a,0}(u_n) \rightarrow \tilde{\theta}_0 \text{ and } I'_{a,0}(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where

$$\tilde{\theta}_0 := \inf \{I_{a,0}(u) : u \in \overline{B}_{R_h}\} < 0, \quad (31)$$

and  $I_{a,0} = I_{a,h}$  for  $h \equiv 0$ . Since  $\{u_n\}$  is a bounded sequence, similar argument to the proof of Theorem 3.3, there exist a subsequence  $\{u_n\}$  and  $u_0^- \in B_{R_h}$  such that  $u_n \rightarrow u_0^-$  strongly in  $H^1(\mathbb{R}^N)$ , which implies that  $I'_{a,0}(u_0^-) = 0$  and  $I_{a,0}(u_0^-) = \tilde{\theta}_0 < 0$ . This yields a nontrivial solution  $u_0^-$  of Equation  $(K_{a,h})$ . Moreover, by Theorem 3.3, there exists a nontrivial solution  $u_0^+$  of Equation  $(K_{a,h})$  such that  $I_{a,0}(u_0^+) = \alpha > 0$ . Therefore,

$$I_{a,0}(u_0^-) = \tilde{\theta}_0 < 0 < \alpha = I_{a,0}(u_0^+)$$

which implies that  $u_0^- \neq u_0^+$ . This completes the proof.

## 4 Proof of Theorem 1.2

For  $h \in L^{2/(2-q)}(\mathbb{R}^N)$ , we define

$$\tilde{\theta}_h := \inf\{I_{a,h}(u) : u \in H^1(\mathbb{R}^N)\}.$$

Then we have the following result.

**Theorem 4.1** *Suppose that conditions  $(D_1) - (D_4)$  and  $(D_6)$  hold. Then there exist  $\bar{a}_0, \bar{\Lambda}_0 > 0$  such that for every  $a \in (0, \bar{a}_0)$  and  $h \in L^{2/(2-q)}(\mathbb{R}^N)$  with  $h \geq 0$  and  $0 < |h|_{L^{2/(2-q)}} < \bar{\Lambda}_0$ , we have*

$$\tilde{\theta}_h \leq \tilde{\theta}_0 < \theta_h < 0,$$

where  $\theta_h$  is as in (33) and  $\tilde{\theta}_0$  is as in (31). Furthermore, there exists a nontrivial solution  $u_{h,2}^-$  of Equation  $(K_{a,h})$  such that  $I_{a,h}(u_{h,2}^-) = \tilde{\theta}_h$ .

**Proof.** If  $h \equiv 0$ , using Theorem 1.1 (ii), then there exists a nontrivial solution  $u_0^-$  of Equation  $(K_{a,h})$  such that

$$I_{a,0}(u_0^-) = \tilde{\theta}_0 = \inf\{I_{a,0}(u) : u \in H^1(\mathbb{R}^N)\} < 0,$$

where  $I_{a,0} = I_{a,h}$  for  $h \equiv 0$ . Assume that  $h \in L^{2/(2-q)}(\mathbb{R}^N) \setminus \{0\}$  with  $h \geq 0$ . Then

$$I_{a,h}(u_0^-) = \tilde{\theta}_0 - \int_{\mathbb{R}^N} h(x) |u_0^-|^q dx < 0.$$

By Lemma 2.7, there exists  $R_* > 0$  such that  $I_{a,h}(u) \geq 0$  for all  $u \in H^1(\mathbb{R}^N)$  with  $\|u\| \geq R_*$ , and

$$\begin{aligned} \tilde{\theta}_h &= \inf\{I_{a,h}(u) : u \in H^1(\mathbb{R}^N)\} \\ &= \inf\{I_{a,h}(u) : u \in \bar{B}_{R_*}\} \leq \tilde{\theta}_0 < 0. \end{aligned}$$

Moreover, by the Ekeland variational principle, there exists a minimizing sequence  $\{u_n\} \subset B_{R_*}$  such that

$$I_{a,h}(u_n) \rightarrow \tilde{\theta}_h \text{ and } I'_{a,h}(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since  $\{u_n\}$  is a bounded sequence, similar argument to the proof of Theorem 3.3, there exist a subsequence  $\{u_n\}$  and  $u_{h,2}^- \in B_{R_*}$  such that  $u_n \rightarrow u_{h,2}^-$  strongly in  $H^1(\mathbb{R}^N)$ . This implies that  $I'_{a,h}(u_{h,2}^-) = 0$  and  $I_{a,h}(u_{h,2}^-) = \tilde{\theta}_h$ . This shows that  $u_{h,2}^-$  is a nontrivial solution of Equation  $(K_{a,h})$ .

Next, we show that  $\tilde{\theta}_0 < \theta_h$ . By Lemma 2.5, if  $h \equiv 0$ , then there exists  $\rho_0 > \rho$  such that  $I_{a,0}(u) > 0$  for all  $u \in B_{\rho_0} \setminus \{0\}$  and  $\inf_{u \in B_{\rho_0}} I_{a,0}(u) = 0$ . Thus, using (33), we can conclude that  $\theta_h \rightarrow 0$  as  $|h|_{L^{2/(2-q)}} \rightarrow 0$ . Thus, there exists  $\bar{\Lambda}_0 > 0$  such that for every

$h \in L^{2/(2-q)}(\mathbb{R}^N)$  with  $h \geq 0$  and  $0 < |h|_{L^{2/(2-q)}} < \bar{\Lambda}_0$ , we have  $\tilde{\theta}_0 < \theta_h$ . This completes the proof. ■

**Now we give the proof of Theorem 1.2:** (i) Since  $h \in L^{2/(2-q)}(\mathbb{R}) \setminus \{0\}$  with  $0 < |h^+|_{L^{2/(2-q)}} < \Lambda_0$ , we can choose a function  $\phi \in H^1(\mathbb{R}^N) \setminus \{0\}$  such that

$$\int_{\mathbb{R}^N} h(x) |\phi^+|^q dx > 0.$$

For  $t > 0$ , we have

$$\begin{aligned} I_{a,h}(t\phi) &= \frac{a}{2} \widehat{m}(\|t\phi\|^2) + \frac{bt^2}{2} \|\phi\|^2 - \int_{\mathbb{R}^N} G(x, t\phi) dx - t^q \int_{\mathbb{R}^N} h(x) |\phi^+|^q dx \\ &\leq \frac{a}{2} \widehat{m}(\|t\phi\|^2) + \frac{t^2}{2} \|\phi\|^2 - \frac{t^2}{2} \int_{\mathbb{R}^N} p_1 \phi^2 dx - t^q \int_{\mathbb{R}^N} h(x) |\phi^+|^q dx \\ &< 0 \end{aligned} \tag{32}$$

for  $t > 0$  small enough. Moreover, by Lemma 2.5, there exists  $\tilde{R}_h \leq \rho$  such that

$$I_{a,h}(u) \geq 0 \text{ for all } u \text{ with } \|u\| = \tilde{R}_h.$$

Hence,

$$\theta_h := \inf\{I_{a,h}(u) : u \in \bar{B}_{\tilde{R}_h}\} < 0. \tag{33}$$

By the Ekeland variational principle, there exists a minimizing sequence  $\{u_n\} \subset B_{\tilde{R}_h}$  such that  $I_{a,h}(u_n) \rightarrow \theta_h$  and  $I'_{a,h}(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\{u_n\}$  is a bounded sequence, similar argument to the proof of Theorem 3.3, there exists a subsequence  $\{u_n\}$  and  $u_{h,1}^- \in B_{\tilde{R}_h}$  such that  $u_n \rightarrow u_{h,1}^-$  strongly in  $H^1(\mathbb{R}^N)$ , which implies that  $I'_{a,h}(u_{h,1}^-) = 0$  and  $I_{a,h}(u_{h,1}^-) = \theta_h < 0$ . This shows that  $u_{h,1}^-$  is a nontrivial solution of Equation  $(K_{a,h})$ .

(ii) By part (i) and Theorem 3.3, there exist two nontrivial solutions  $u_{h,1}^-$  and  $u_h^+$  of Equation  $(K_{a,h})$  such that

$$I_{a,h}(u_{h,1}^-) = \theta_h < 0 < \alpha = I_{a,h}(u_h^+),$$

where  $\theta_h$  is as in (33). This shows that  $u_{h,1}^- \neq u_h^+$ .

(iii) By part (ii) and Theorem 4.1, Equation  $(K_{a,h})$  has three nontrivial solutions  $u_{h,1}^-$ ,  $u_{h,2}^-$  and  $u_h^+$  with

$$\tilde{\theta}_h = I_{a,h}(u_{h,2}^-) < I_{a,h}(u_{h,1}^-) = \theta_h < 0 < \alpha = I_{a,h}(u_h^+),$$

which implies that  $u_{h,1}^-$ ,  $u_{h,2}^-$  and  $u_h^+$  are different. This completes the proof.

## 5 Proof of Theorem 1.4

**Now we give the proof of Theorem 1.4:** Let  $u$  be a nontrivial solution of Equation  $(K_{a,h})$ . Then we have

$$\begin{aligned} 0 &= \langle I'_{a,b}(u), u \rangle \\ &= am(\|u\|^2) \|u\|^2 + b \|u\|^2 - \int_{\mathbb{R}^N} g(x, u) u dx - \int_{\mathbb{R}^N} h |u^+|^q dx. \end{aligned}$$

The condition  $(D_7)$  implies that

$$\int_{\mathbb{R}^N} g(x, u) u dx \leq \int_{\mathbb{R}^N} p_2 u^2 dx.$$

Thus, by (4) and the condition  $(D_8)$ ,

$$\begin{aligned} 0 &\geq am(\|u\|^2) \|u\|^2 + b \|u\|^2 - \int_{\mathbb{R}^N} p_2 u^2 dx - \int_{\mathbb{R}^N} h |u^+|^q dx \\ &\geq am(\|u\|^2) \|u\|^2 - \frac{1 - b\mu^*}{\mu^*} \|u\|^2 - \frac{|h|_{L^{2/(2-q)}}}{S_2^q} \|u\|^q \\ &> 0, \end{aligned}$$

which is a contradiction. Therefore, Equation  $(K_{a,h})$  does not admits any nontrivial solution. This completes the proof.

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